INVESTIGATING THE STABILITY OF SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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Some criteria of existence of a sector for a system of equations of perturbed motion are cited. These criteria are then used as a basis for deriving several new theorems falling within the context of the second method of Liapunov.

Let us consider the system of differential equations

$$dx_s / dt = f_s(t, x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{1}$$

whose right sides are continuous in the domain

$$(h)\; t \geqslant 0, \; \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \leqslant A$$

and $f_s(t, 0, 0, \ldots, 0) \equiv 0$.

We shall use the symbols $\alpha_1, \ldots, \alpha_n$ to denote quantities of which each one can assume either of the two values 1, −1.

Let the indicated parameters take on some fixed values $\alpha_s = \alpha_{s0}$, $(s = 1, 2, \ldots, n)$ and let us denote by $K_0 \{\alpha_{10}, \ldots, \alpha_{n0}\}$ the set of all those points $(t, x_1, \ldots, x_n) \in h$ for which none of the coordinates $x_s \neq 0$ and

$$\operatorname{sign} x_s = \alpha_{s0} \qquad (s = 1, 2, \ldots, n) \tag{2}$$

We shall say that the numbers $\alpha_{10}, \ldots, \alpha_{n0}$ themselves form the basis of the region K_0 under consideration.

The set $\sigma \{\alpha_{10}, \ldots, \alpha_{n0}\}$ of all those boundary points of the region $K_0 \{\alpha_{10}, \ldots, \alpha_{n0}\}$ for which one or several coordinates $x_s = 0$ shall be called a side surface of this region and we set

$$K \{\alpha_{10}, \ldots, \alpha_{n0}) = K_0 \{\alpha_{10}, \ldots, \alpha_{n0}\} \; U \; \sigma \{\alpha_{10}, \ldots, \alpha_{n0}\} \tag{3}$$

If the domain h is defined by the inequalities

$$t \geqslant 0, \qquad \|x\| < \infty \tag{4}$$

then the set $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ associated with some basis $\{\alpha_{s0}\}$ is a "cone".

Definition 1. We say that the right sides of system (1) in the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ "have the property of preserving the signs of the elements of the basis $\{\alpha_{s0}\}$" if the following inequalities are fulfilled at the points of the side surface $\sigma \{\alpha_{10}, \ldots, \alpha_{n0}\}$:

$$\alpha_{s0} f_s(t, x_1, \ldots x_{s-1}, 0, x_{s+1}, \ldots x_n) \geqslant 0 \qquad (s = 1, 2, \ldots, n) \tag{5}$$

Definition 2. Let $V(t, x_1, \ldots, x_n)$ be some Liapunov function. We call this function "positive-definite" in the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ if there exists a function $\omega(x_1, \ldots x_n)$ independent of t and positive-definite in the domain h such that the inequality $v \geqslant \omega$ is fulfilled at all points $(t, x_1, \ldots, x_n) \in K \{\alpha_{10}, \ldots, \alpha_{n0}\}$.

The proof of the following statement is similar to the proof of Lemma 4.1 in [1].

Lemma 1. Let the right side of system (1) in the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ have the property of preserving the signs of the elements of the basis $\{\alpha_{s0}\}$.

At least one solution of this system which for all $t \geqslant t_0$ either remains within the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ or can leave it only by way of points of the surface

$$t \geqslant 0, \qquad \| x \| = A \tag{6}$$

then passes through any point $(t_0, x_{10}, \ldots, x_{n0}) \in K_0 \{\alpha_{10}, \ldots, \alpha_{n0}\}$.

Lemma 2. If the conditions of Lemma 1 are fulfilled, if the domain h is defined by inequality (4), and if the solutions of system (1) are unique, then the corresponding domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ is a positively invariant set for this system.

For example, in the case of the system of linear equations

$$dx_s/dt = P_{s1}(t) x_1 + \ldots + p_{sn}(t) x_n \quad (s = 1, 2, \ldots, n) \tag{7}$$

with continuous coefficients the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ is a positively invariant set if the basis $\{\alpha_{s0}\}$ and the coefficients of the system are related by the expressions

$$p_{ks}\alpha_{s0}\alpha_{k0} \geqslant 0 \quad \text{for } s \neq k \qquad (s, k = 1, 2, \ldots, n) \tag{8}$$

for all values of $t \geqslant 0$.

It is easy to see that the cone $K \{-\alpha_{10}, \ldots, -\alpha_{n0}\}$ is also a positively invariant set in this case.

Specifically, if $p_{sk}(t) \geqslant 0$ for $s \neq k$, then system (7) has the two positively invariant sets $K \{1, 1, \ldots, 1\}$ and $K \{-1, -1, \ldots, -1\}$; this result agrees with [1].

We note that fulfillment of the conditions of Lemma 1 means that the corresponding domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ is a sector [2], so that this lemma can be used to construct certain criteria of instability.

For example, we have the following theorems.

Theorem 1. Let the right sides of system (1) in the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ have the property of preserving the signs of the elements of its basis, and let the inequality

$$\sum_{s=1}^{n} \alpha_{s0} A_s f_s (t, x_1, \ldots, x_n) \geqslant \lambda(t) W(t, x_1, \ldots, x_n)$$

where

$$\lambda(t) \geqslant 0, \qquad \lim_{t \to \infty} \int_0^t \lambda(\tau)\, d\tau = \infty$$

be fulfilled at points of the domain $K \{\alpha_{10}, \ldots, \alpha_{n0}\}$ for certain real constants $A_1, \ldots, A_n$ among at least one $A_s > 0$. The function W in the above inequality is positive-definite in the domain in question.

The zero solution of system (1) is then unstable.

Theorem 2. Let the elements of some basis $\{\alpha_{s0}\}$ and the coefficients of system (7) be related by expressions (8) for $t \geqslant 0$.

The existence of real constants $c_1, \ldots, c_n$ (at least one of which is a number c_l larger than zero) satisfying the inequalities

$$\sum_{\substack{k=1 \\ (k \neq s)}}^{n} | p_{ks}(t) | c_k + p_{ss}(t) c_s \geqslant F(t) \geqslant 0 \qquad (s = 1, 2, \ldots, n) \tag{9}$$

where

$$\lim_{t \to \infty} \int_0^t F(\tau)\, d\tau = \infty$$

then implies that the zero solution of the system in question is unstable.

Theorem 3. If the elements of some basis $\{\alpha_{s0}\}$ and the coefficients of system (7) satisfy inequalities (8) for $t \geqslant 0$ and if the relation

$$\lim_{t\to\infty}\int_0^t p_{ss}(\tau)\,d\tau=\infty \tag{10}$$

is fulfilled for at least one of the diagonal coefficients $p_{ss}(t)$ of this system, then the zero solution of system (7) is unstable.

These theorems can be readily proved with the aid of Liapunov functions expressed as certain linear forms.

For example, to prove Theorem 3 we can set

$$v=\alpha_{s0}x_s\exp\left(-\int_0^t p_{ss}(\tau)\,d\tau\right) \tag{11}$$

The function v is then larger than zero at the points of the set $K_0\{\alpha_{10},\ldots,\alpha_{n0}\}$, and its total derivative in the domain $K\{\alpha_{10},\ldots,\alpha_{n0}\}$ satisfies the inequality

$$v'=\alpha_{s0}\exp\left(-\int_0^t p_{ss}(\tau)\,d\tau\right)(p_{s1}(t)x_1+\ldots+p_{s,s-1}(t)x_{s-1}+$$
$$+p_{s,s+1}(t)x_{s+1}+\ldots+p_{sn}(t)x_n)\geqslant 0$$

by virtue of system (7).

Thus, all of the conditions of a certain theorem on instability with a sector formulated in [3] are fulfilled for the system just considered here.

Now let us consider the system of linear differential equations

$$dx_s/dt=p_{s1}x_1+\ldots+p_{sn}x_n \qquad (s=1,2,\ldots,n) \tag{12}$$

whose coefficients are real constants.

Lemma 3. Let the coefficients of system (12) and the elements of some basis $\{\alpha_{s0}\}$ be related by expressions (8).

Then all the roots of the secular equation

$$\det\|p_{sk}-\lambda\delta_{sk}\|=0 \tag{13}$$

have negative real parts if and only if all the numbers $b_1,\ldots,b_n$ determined from the system

$$\sum_{k=1}^{n}p_{ks}\alpha_{k0}b_k=-\alpha_{s0}a_s \qquad (s=1,2,\ldots,n) \tag{14}$$

are positive for all positive $a_1,\ldots,a_n$.

Necessity. Let all the roots of Eq. (13) have negative real parts. Then the determinant of system (14)

$$\Delta=\alpha_{10}\ldots\alpha_{n0}\det\|p_{sk}\|\neq 0$$

Let us set

$$v(x_1,\ldots,x_n)=\sum_{s=1}^{n}b_s\alpha_{s0}x_s$$

From (14) it follows that none of the numbers $b_s\neq 0$. Let us assume that at least one of the quantities $b_s<0$. Then the function v would satisfy, by virtue of system (12), all conditions of Liapunov's first instability theorem at points of positively invariant set $K\{\alpha_{10},\ldots,\alpha_{n0}\}$ while the null solution of this system is asymptotically stable.

Sufficiency and the following lemma are proved analogously.

Lemma 4. Let the coefficients of system (12) be such that inequalities (8) are fulfilled for some basis $\{\alpha_{s0}\}$ and $\det\|p_{sk}\|\neq 0$.

Equation (13) then has at least one root with a positive real part or roots with real parts equal to zero such that the number of groups of solutions corresponding to these roots is smaller than their multiplicity if and only if there exists at least one negative number b_s determined from system (14).

If relations (8) are fulfilled, the system (14) can be written as

$$\sum_{\substack{k=1 \\ (k \neq s)}}^{n} |p_{ks}| b_k + p_{ss} b_s = -a_s \qquad (s = 1, 2, \ldots, n) \tag{15}$$

The above lemmas readily yield several theorems on stability and instability for the system of equations

$$dx_s / dt = p_{s1}\varphi_1(t, x_1, \ldots, x_n) + \ldots + p_{sn}\varphi_n(t, x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, n) \tag{16}$$

where p_{sk} are real constants, where the functions φ_s are continuous in h, and where $\varphi_s(t, 0, 0, \ldots, 0) \equiv 0$.

Theorem 4. Let the following conditions be fulfilled in the domain $K\ \{\alpha_{10}, \ldots, \ldots, \alpha_{n0}\}$:

1) the right sides of system (16) have the property of preserving the signs of the elements of the basis $\{\alpha_{s0}\}$;

2) for certain positive constants $A_1 \ldots, A_n$,

$$\sum_{s=1}^{n} A_s \alpha_{s0} \varphi_s(t, x_1 \ldots, x_n) \geqslant \lambda(t) W(t, x_1, \ldots, x_n) \tag{17}$$

where

$$\lambda(t) \geqslant 0, \qquad \lim_{t \to \infty} \int_0^t \lambda(\tau)\, d\tau = \infty$$

and W is a positive-definite function in the domain $K\ \{\alpha_{10}, \ldots, \alpha_{n0}\}$;

3) the elements of the basis $\{\alpha_{s0}\}$ and the coefficients of the system are related by expressions (8).

If Eq. (13) does not have roots equal to zero but has either at least one root with a positive real part or roots with real parts equal to zero such that the number of groups of solutions corresponding to these roots is smaller than the multiplicity of the latter, then the zero solution of system (16) is unstable.

To prove the theorem we determine the constants $B_1, \ldots, B_n$ from the system of equations

$$\sum_{k=1}^{n} p_{ks} \alpha_{k0} = \alpha_{s0} A_s \qquad (s = 1, 2, \ldots, n) \tag{18}$$

and set

$$V(x_1, \ldots, x_n) = \sum_{s=1}^{n} B_s \alpha_{s0} x_s \tag{19}$$

Lemma 4 implies that at least one of the numbers $B_s > 0$, so that the form V is able to assume positive values in the domain $K\ \{\alpha_{10}, \ldots, \alpha_{n0}\}$; moreover,

$$V' = \sum_{s=1}^{n} \alpha_{s0} A_s \varphi_s(t, x_1, \ldots, x_n) \geqslant 0 \tag{20}$$

at the points of this domain.

Let us suppose that the zero solution of system (16) is stable. Then for any number $\varepsilon > 0$ $(\varepsilon < A)$ there exists a number $\delta > 0$ such that none of the integral lines of system (16) which lie on the sphere $\|x\| = \delta$ for $t = 0$ reach the sphere $\|x\| = \varepsilon$ for any $t \geqslant 0$.

Let us choose a point $(0, x_1, \ldots, x_n) \in K_0\{\alpha_{10}, \ldots, \alpha_{n0}\}$ on the sphere $\|x\| = \delta$ such that $V > 0$ and consider the integral line of system (16), namely

$$x_s = u_s(t) \qquad (s = 1, 2, \ldots, n) \tag{21}$$

which passes through this point and (by Lemma 1) does not intersect the side surface $\sigma\{\alpha_{10}, \ldots, \alpha_{n0}\}$ for any $t \geqslant 0$. We infer from (20) that the integral line in question does not have points in common with a certain sufficiently small neighborhood h_α of the origin defined by the inequalities $t \geqslant 0$, $\|x\| \leqslant \alpha$, $(0 < \alpha < \delta)$. But the function $W(t, x_1, \ldots, x_n) \geqslant \beta > 0$ (where β is some sufficiently small number) at the points of the set $K\{\alpha_{10}, \ldots, \alpha_{n0}\} \setminus h_\alpha$. This implies that the inequality

$$V(u_1(t), \ldots, u_n(t)) \geqslant V(x_{10}, \ldots, x_{n0}) + \beta \int_0^t \lambda(\tau)\, d\tau$$

must be fulfilled along solution (21) for all $t \geqslant 0$. The latter inequality is definitely invalid for sufficiently large t.

Let us be given a system of equations

$$dx_s/dt = p_{s1}\varphi_1(x_1, \ldots, x_n) + \cdots + p_{sn}\varphi_n(x_1, \ldots, x_n) + R_s(x_1, \ldots, x_n)$$
$$(s = 1, 2, \ldots, n) \tag{22}$$

where p_{sk} are real constants, φ_s are polynomials in the quantities $x_1, \ldots, x_n$ of degree not higher than $N \geqslant 1$, and $\varphi_s(0, \ldots, 0) = 0$; the functions R_s can be expanded in some neighborhood of the origin in powers of $x_1, \ldots, x_n$ (the leading terms of the series are of order not lower than $N + 1$).

Theorem 5. Let the right sides of system (22) in the domain $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ have the property of preserving the signs of the elements of the basis $\{\alpha_{s0}\}$ and let them satisfy the following conditions:

1) the function

$$U(x_1, \ldots, x_n) = \sum_{s=1}^{n} A_s \alpha_{s0} \varphi_s(x_1, \ldots, x_n) \tag{23}$$

is a positive-definite form in the domain $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ for some positive constants $A_1, \ldots, A_n$,

2) the coefficients p_{sk} and the elements of the basis $\{\alpha_{s0}\}$ are related by expressions (8).

Then, if $\det \|p_{sk}\| \neq 0$ and if Eq. (13) has either at least one root with a positive real part or roots with real parts equal to zero such that the number of groups of solutions corresponding to these roots is smaller than their multiplicity, then the zero solution of system (22) is unstable.

We can prove this theorem simply by noting that by virtue of system (22) the total derivative of the linear form (19) is a positive-definite function at the points of intersection of the set $K\{\alpha_{10}, \ldots, \alpha_{n0}\}$ with some sufficiently small neighborhood of the origin.

Theorem 6. Let the functions φ_s in the domain h satisfy the conditions

$$\varphi_s(t, x_1, \ldots, x_n) \operatorname{sign} x_s \geqslant 0 \qquad (s = 1, 2, \ldots, n) \tag{24}$$

The existence of positive constants $A_1, \ldots, A_n$ satisfying the inequalities

$$\sum_{\substack{k=1 \\ (k \neq s)}}^{n} |p_{ks}| A_k + A_s p_{ss} \leqslant 0 \qquad (s = 1, 2, \ldots, n) \tag{25}$$

then implies that the zero solution of system (16) is stable.

In fact let us set

$$V(x_1, \ldots, x_n) = \sum_{s=1}^{n} A_s |x_s| \tag{26}$$

It is easy to see from system (16) that

$$V' \leqslant \sum_{s=1}^{n} \Big(\sum_{k=1}^{n} |p_{ks}| A_k + p_{ss} A_s \Big) \varphi_s(t, x_1, \ldots, x_n) \operatorname{sign} x_s \leqslant 0 \tag{27}$$

which proves the theorem.

We note that relations (25) are strict inequalities when the coefficients p_{sk} and the elements of some basis $\{\alpha_{s0}\}$ are related by expressions (8) and that all the roots of Eq. (13) have negative real parts.

We also note that Theorem 6 is a modification and refinement of a certain theorem on stability formulated in our paper [4].

The following criterion of asymptotic stability is analogous to Theorem 6 and is closely related to one of the theorems of [5].

T h e o r e m 7. If fulfillment of the conditions of Theorem 6 turns expressions (25) into strict inequalities and if the function

$$U(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} \varphi_s(t, x_1, \ldots, x_n) \operatorname{sign} x_s \tag{28}$$

is positive-definite, then the zero solution of system (16) is asymptotically stable and uniform in t_0 and x_{s0}.

Let us consider two corollaries of this theorem.

C o r o l l a r y 1. Let system (22) be such that the functions $\varphi_s(x_1, \ldots, x_n)$ satisfy conditions (24). If the expression

$$\Phi(x_1, \ldots, x_n) = \sum_{s=1}^{n} \varphi_s(x_1, \ldots, x_n) \operatorname{sign} x_s \tag{29}$$

is a homogeneous positive-definite function and if relations (25) are strict inequalities for some positive $A_1, \ldots, A_n$, then the zero solution of system (22) is asymptotically stable.

C o r o l l a r y 2. Let the domain of definition h of the right sides of system (16) be given by inequalities (4).

If all the conditions of Theorem 7 are fulfilled in this domain, the zero solution of system (16) is asymptotically stable in the large and uniform in t_0 and x_{s0}.

We note that by virtue of the system of differential equations under consideration, function (26) in this case satisfies all the conditions of the theorem on uniform asymptotic stability in the large formulated in [6].

Finally, let us consider as an example the system of differential equations

$$dx_s/dt = p_{s1}\varphi_1(x_1) + \ldots + p_{sn}\varphi_n(x_n) \quad (s = 1, 2, \ldots, n) \tag{30}$$

where P_{sk} are real constants and $\varphi_s(x_s)$ are continuous and satisfy the inequalities.

$$\varphi_s(x_s)\,\mathrm{sign}\, x_s > 0 \quad \text{for } x_s \neq 0 \qquad (s = 1, 2, \ldots, n) \tag{31}$$

Let the coefficients of this system and the elements of some basis $\{\alpha_{s\theta}\}$ be related by expressions (8). We then draw the following conclusions on the basis of Theorem 4 and Corollary 2 of Theorem 7:

1) The system under consideration is absolutely stable if and only if all the roots of secular equation (13) have negative real parts.

2) Let fulfillment of the above assumptions concerning the right sides of system (30) imply that $\det \|p_{sk}\| \neq 0$. The zero solution of this system is then unstable for any chosen functions $\varphi_s(x_s)$ satisfying inequalities (31) if and only if: (a) there exists at least one root of Eq. (13) with a positive real part, or (b) there exist roots of this equation with real parts equal to zero such that the number of groups of solutions corresponding to these roots is smaller than their multiplicity.

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ON THE STABILITY OF TRIANGULAR LIBRATION POINTS IN THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM

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The results of a study of the stability of the equilibrium position of a nonautonomous Hamiltonian system with two degrees of freedom are presented. The parametric resonance domain for the libration points is determined to within the first power of the eccentricity. Formulas for computing the characteristic exponents are derived. The resonance values of μ and e for which the libration points can be unstable inside the stability domains are determined.

1. Let us consider three material points which attract each other according to Newton's law. Let the points S and J of masses m_1 and m_2 move relative to their common